Project 2: How Parentheses and the Order of Operations Impose Structure on Expressions

Parentheses show us how things should be grouped together. The sole purpose of parentheses in algebraic expressions is to impose a particular structure, where certain things are grouped together. Things together within a set of parentheses are treated as a single unit. So, for example:

- $(2x+1)-(3x+7)$ Here we treat $(2x+1)$ and $(3x+7)$ each as single units, the whole expression $(3x+7)$ is being subtracted from the whole first expression $(2x+1)$.
- $3(xy)^3$ Here we treat (xy) as a single unit. The whole expression (xy) is being raised to the third power.
- $3(2x^2-3x+1)$ Here we treat $(2x^2-3x+1)$ as a single unit. The whole expression $(2x^2-3x+1)$ is being multiplied by 3.

Implied or hidden parentheses (and other hidden symbols)

There are a number of conventions for writing algebraic expressions and equations that allow us to leave out parentheses or other symbols. Conventions are just rules that mathematicians have made up to allow us to write certain things quickly or simply because they are like abbreviations—we can write something using fewer parentheses or symbols, and this makes the expressions faster to write and easier to read. However, in order for this to work, we do have to memorize the conventions so that we know how to read mathematical expressions correctly. Here are some examples of some common math conventions that we need to know:

I. Two things written next to each other (except for two numbers) means multiplication. So, for example:

- $-2x=-2\cdot x$
- $pq=p\cdot q$
- $2ab^3c=2\cdot a\cdot b^3\cdot c$
- $(3x^2-4)(x^2+5x-9)=(3x^2-4)\cdot(x^2+5x-9)$
- $3\sqrt{4}=3\cdot\sqrt{4}$
- $(2)(3)=(2)\cdot(3)$
  Note: 23 is NOT two times three, but rather the number twenty-three. The parentheses here do NOT mean multiplication—the fact that the two factors are written next to each other is what indicates multiplication. However, the parentheses are included here to make clear that this is 2 times 3 and NOT the number 23.

Now you try! Put in the missing multiplication sign (use a dot instead of an x):
For each of the following questions, rewrite the expression using the multiplication dot in between each pair of factors that are being multiplied together (you can use the examples above as a guide):

1. xy
2. $4p$
3. $2\sqrt{3}$
4. $(-2)(7)$
5. $(2x-1)(3x^2+7)$
6. $-3pqr^2$

II. Subtraction: the minus sign applies to whatever term directly follows it. If a parenthesis is just to the right of the minus sign, then everything inside the parentheses must be subtracted. If a term that begins with a number or variable is to the right of the minus sign, then only that term is subtracted. So, for example:

- $(2x+1)-(3x+7)$ The whole $(3x+7)$ is subtracted.
- $3x^2-2x+4x+5=3x^2-(2x)+4x+5$ Just the $2x$ is subtracted. If we want to put a parenthesis in front of the $2x$, we MUST close the parenthesis IMMEDIATELY AFTER the $2x$. We CANNOT, for example, put a set of parentheses around all of the terms $2x+4x+5$, because then we would be subtracting ALL of these terms—but in this problem, we are ONLY supposed to be subtracting the $2x$

Now you try! Put in the missing parentheses:
For each of the following questions, rewrite the expression by putting parentheses around whatever is actually being subtracted. If parentheses are already around what is being subtracted, CIRCLE the part that is being subtracted (you can use the examples above as a guide):

7. $3x^2+7-2x+1$

8. $(3x^2+7)-(2x+1)$

9. $2-xyz$

10. $(2-x)yz$

III. Exponents only apply to whatever is directly to the left of the exponent. If a parenthesis is just to the left of the exponent, then everything inside the parentheses is the base. If a number or variable is to the left of the exponent, then only that number or variable is the base—if there is a negative sign out front, it is NOT part of the base, unless the whole thing is in parentheses. So, for example:

- $-2^2=-(2)^2$ Only 2 is squared. (So we would get: $-(2\cdot 2)=-4$.)
- $(-2)^2$ Negative 2 is squared. (So we would get: $(-2)(-2)=-4$.)
- $3xy^3=3x(y)^3$ Only y is raised to the third power. (So this is really $3xyyy$.)
- $3(xy)^3$ The whole expression xy is raised to the third power, but 3 is not. (So this is really $3xyxyxy$.)
- $(3xy)^3$ The whole expression $3xy$ is raised to the third power. (So this is really $(3xy)(3xy)(3xy)$.)
- $x-y^2=x-(y)^2$ Only y is squared, not the x, and not the negative sign. (So this is really $x-(y)(y)$.)
- $(x-y)^2$ The whole expression $x-y$ is squared. (So this is really $(x-y)(x-y)$.)

Now you try! Put in the missing parentheses:
For each of the following questions, rewrite the expression by putting parentheses around whatever is actually being raised to the exponent. If parentheses are already around what is being raised to the exponent, CIRCLE the part that is being raised to that exponent (you can use the examples above as a guide):

11. $-4x+7^3$

12. $(-4x+7)^3$

13. $-4x^3+7$

14. $(-4x)^3+7$

15. $-(4x)^3+7$

16. -4^3x+7

17. $(-4)^3x+7$

18. $-4(x+7)^3$

IV._Fraction bars and radial signs function like parentheses. For fractions, the top and bottom of the fraction are really inside their own parentheses, even though we don't write them out. Anything under a radical sign is really inside parentheses, even though we don't write that out either. So, for example:

- $\frac{2-x}{x}=\frac{(2-x)}{x}$ or $(2-x)/x$ or $(2-x)\div x$
- $\frac{3a^2+b}{a-b}=\frac{(3a^2+b)}{(a-b)}$ or $(3a^2+b)/(a-b)$ or $(3a^2+b)\div(a-b)$
- $\sqrt{x^2+y^2}=\sqrt{(x^2+y^2)}$

Now you try! Put in the missing parentheses:
For each of the following questions, rewrite the expression by putting parentheses around whatever is the numerator or denominator of a fraction, or whatever is under a radical. If parentheses are already around one of these things, CIRCLE that part intead (you can use the examples above as a guide):

19. $\frac{-4x+7}{2x}$

20. $\frac{xy^2-xyz}{xy^2-x^2y}$

21. $\sqrt{3xy^2z^3}$

22. $\sqrt[3]{2x^2-y^2}$

How do we group things in an expression? The order of operations imposes structure on expressions

You may have already learned the order of operations as a set of rules that tells us what order to do certain operations in when making calculations with numbers. However, when thinking algebraically, we need to think about the order of operations differently:

1. The order of operations is just a set of arbitrary conventions about where we should assume the hidden parentheses are (just like the conventions listed above); and
2. The order of operations imposes a structure on algebraic expressions, telling us which parts of the expression we should clump together and treat as a single unit.

Aside from what you have learned in the past, there are really only two basic conventions that we need to keep in mind in order to remember the order of operations:

A. **Things inside parentheses are always treated as a single unit, from the inside out.** This includes implied parentheses that we have to assume are there but that aren't written out, like those around the top and bottom of a fraction, or those around what is under a radical sign.
B. **The order in which we should group things otherwise (if there are no parentheses imposing a different order) is always the following:**

1. **Exponents/multiplication/division (make sure all division is written in fraction form first)**
2. **Addition/subtraction (rewrite all subtraction as adding a negative first)**

How does this play out in actual expressions? Consider the following examples:

- $2x^2-(xy)^4$ Here there are several structures. Things written next to each other indicate multiplication, so first we put in the multiplication dots: $2\cdot x^2-(x\cdot y)^4$. Then, before we can deal with the subtraction, we need to change it to adding a negative, so this gives us: $2\cdot x^2+-(x\cdot y)^4$. The parentheses around the xy tell us that the whole expression xy (but NOT the minus sign) is being raised to the fourth power, so we can add another set of parentheses to make this clear: $2\cdot x^2+-((x\cdot y)^4)$. The fact that the x in the first term is to the left of the exponent tells us that only the x is being squared. So we can rewrite this as $2\cdot(x)^2+-((x\cdot y)^4)$. And finally, we know that we have to multiply 2 by $(x)^2$ before we can add the two terms, so we can add another set of parentheses to show this: $(2\cdot(x)^2)+-((x\cdot y)^4)$. This expression looks a lot messier, but all we have done is to put in all the hidden parentheses that tell us how the different things in the expression are actually grouped. You could also do this by using circles instead of parentheses, if that is clearer, something like this: $\boxed{2\cdot\boxed{x}^2}+-\boxed{\boxed{x\cdot y}^4}$.
- $\frac{3(xy^2)^3}{2x-xy}$ Similarly, here the y is the only thing being squared, so we put parentheses around the y; then the whole $(x(y)^2)$ term is being raised to the third power, so we recognize that the parenthesis that are already there impose this structure; we add in the hidden multiplication dots for things written next to each other; then we notice that the top and the bottom of the fraction imply that there are parentheses around the top and the bottom, so we put them in; the subtraction in the bottom can be rewritten as adding a negative, so we do that; the multiplication in the first and last term in the bottom of the fraction needs to be done before the addition, so we put parentheses around each of these terms, which yields: $\frac{\left(3\cdot(x\cdot(y)^2)^3\right)}{((2\cdot x)+-(x\cdot y))}$ or if we use circles/boxes instead of parentheses it would look like this: $\dfrac{\boxed{3\cdot\boxed{x\cdot\boxed{y}^2}}^3}{\boxed{\boxed{2\cdot x}+-\boxed{x\cdot y}}}$.

__**Now you try! Put in the hidden parentheses (and other hidden symbols):**__

For each of the following questions, insert parentheses for each of the following (use the examples above as a guide, and feel free to use circles instead of parentheses if it is easier for you to see the structure that way):

- **To indicate exactly what term is being subtracted, for each minus sign**
- **To indicate the base of each exponent**
- **To indicate the top and bottom of each fraction**
- **To indicate what is under the radical sign**
- **To indicate the order in which we should group things, based on the order of operations**
- **Also, add any missing multiplication signs, rewrite any addition as adding a negative, and rewrite any division as a fraction**

23. $8x^2-xy$

24. $2x^5y^{-4}$

25. $\frac{2x^2-4x}{2x}$

26. $\sqrt{xy^2}$

27. $3x^2-2x+5x^2$

28. $\sqrt{2}\left(\sqrt{2}+\sqrt{7}\right)$

29. $(3x^2y)(-4xy^3)$

30. $(7x^2-6x+2)-(4x^2-3x+5)$

31. $(4x-2)(x^2-5x+3)$

32. $\frac{20x^9-30x^4+10x^3}{-10x^3}$

Variables can be used to represent general structures

You may be used to seeing properties (patterns that we observe to be true with certain types of numbers or expressions) written out as equations using variables. This is a way in which variables are standing in for certain structures. So, for example, you may have seen the following property:

$$a(b+c) = ab + ac$$

What this equation means, is that any time we see something with the structure $a(b+c)$ we can replace it with the structure $ab + ac$ (but in order to do that we first have to identify what a, b and c are equal to in the particular expression we are working with. But in algebra, a, b and c can stand in for numbers, for other variables like x, or for other more complicated expressions like $-2xy - 5$. Let's look at some examples so that we can get practice identifying the specific structure $a(b+c)$. All of the following expressions have the structure $a(b+c)$ as long as we choose the right values for a, b and c:

- $2(x+4)$ $a = 2,\ b = x,\ c = 4$
- $3(2x-5)$ First we have to rewrite addition as subtraction: $2(2x + -5)$. Then $a = 3, b = 2x, c = -5$
- $\sqrt{2}(\sqrt{2}+\sqrt{7})$ $a = \sqrt{2},\ b = \sqrt{2},\ c = \sqrt{7}$ (Notice that in this case we happen to have $a = b$, which is allowed, although it will not happen most of the time.)
- $3xy(2xy^2 + 3(xy)^3)$ $a = 3xy,\quad b = 2xy^2,\quad c = 3(xy)^3$
- $-2x(3x^2 + 3x + 4)$ First we have to group things inside the parentheses so that we have <u>two</u> terms inside. There are a few different ways to do this, but we could pick this (note that we can only do this because we have three terms that are all being <u>added</u>, and addition can be grouped any way we like): $-2x\big(3x^2 + (3x+4)\big)$. Then we would have $a = -2x,\quad b = 3x^2,\quad c = (3x+4)$
- $(2x-1)(3x+7)$ $a = (2x-1),\quad b = 3x,\quad c = 7$

<u>Now you try! Identify the structure:</u>
For each of the following expressions, look for the structure $(a+b)c$. Notice that this is similar to the examples above, but this is a <u>different</u> structure than those examples. For each one, identify what a, b and c would have to be equal to in order for the expression to have the structure $(a+b)c$:

33. $(3x+2)x$

34. $(\sqrt{3}-\sqrt{5})\sqrt{2}$

35. $(2x-2)2$

36. $(3x^2 + 2xy + y^2)y^2$

37. $(2x+1)(3x-4)$

38. $\left(3xy^2 \quad + 4(x^2y)^3\right) 3x^2y$